From Physics to Economics: the versatility of the exponential function

I. Introduction. General terms and requirements.

After having completed your team and chosen your leader, you should decide on how to share the work to be done. You are free to make any choice you want: each member can work alone on one or two of the four sub-topics, explained in parts III, IV, V and VI below. You can also try to handle one or more sub-topics together, exchanging ideas about possible solutions. In any case your solutions should be correct, clearly stated and thoroughly documented.

In addition to the general evaluation criteria stated in the **STEM4youth Open Student Competition Rules**, the spreadsheets you will elaborate shall also be evaluated. Thus, such features as correctness and simplicity, user-friendliness and clarity, both of the spreadsheet itself and of the solution obtained and presented by its means, will be taken into account.

Your work is divided into four sub-topics, described in parts III, IV, V and VI. Each of them has roughly the same impact on your team's assessment. Similarly, any choice you make in the two sub-topics that offer you such a possibility (parts V and VI), will have the same impact on your team's assessment.

Part II "Basics in mathematics..." gives you a general idea of what is expected in this topic. Get acquainted with it - it will be useful when solving the problems proposed in the further parts.

In part III "Mathematics and finances" you will find four problems based on the story of the two twins (cf. part II). Solve all of them.

Part IV "Physics" contains two problems in physics, related to phenomena described by exponential functions: nuclear decay and charging an electric capacitor. You are expected to present solutions of both problems.

In part V "Mathematical modeling" you are presented a short description of three issues in different fields. After choosing one of them, you are supposed to elaborate a mathematical model of the system and to comment whether or not the obtained solution is an exponential function. Each problem has two additional tasks, also to be solved.

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Part VI "Experimental data analysis" gives you short descriptions of three experiments, each of them with a table of fictitious experimental data. The common feature of all three experiments is that the results resemble - at first sight - an exponential dependence. Your task, in general, will be to verify - for one chosen experiment - whether its results fit an exponential function or not. You may use any appropriate scientific tools to achieve this goal.

Please submit your work in the form of documents in commonly accepted formats (e.g. PDF, Microsoft Office, Open Office). The same applies to spreadsheets and any other media you might use, such as pictures, videos, etc.

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II. Basics in mathematics and mathematical modeling. An exemplary problem: the story of two twins.

1. A process may be "uniform"; this means that a given physical quantity changes by the same amount in equal time intervals. The time dependence of such a quantity is given by a linear function. This also implies the existence of a derived quantity, constant in time, which we interpret as the "speed" of the original quantity. This "speed" is the slope coefficient of the linear function mentioned above.

   A well-known example is the uniform movement. Indeed, in such a movement the position x (that's the original quantity) of a physical body changes by the same amount Δx in equal time intervals Δt. The derived quantity, related to position, is simply the speed v of the body. The dependence of position on time is given by the following linear function:

   $$x(t) = v \cdot t + x_0 \qquad (x_0 \text{ is the body's initial position})$$

   We can write the same using the position change Δx:

   $$\Delta x = v \cdot \Delta t \qquad (1)$$

   This may be considered as a formula for obtaining Δx.

2. In many processes however, the change of a physical quantity depends on the momentary value of this quantity. Such processes are studied not only in physics, chemistry or biology, they also appear in social and economical sciences. The most simple case is when the change of the envisaged quantity is directly proportional to its value. This implies the existence of a coefficient λ used to describe the dependence of the change of the quantity on its momentary value.

   Let's take a look at an example of such a situation in finances. Each one of two twins, Suzan and Jack, got a nice amount of money for their birthday – let's denote it by M_0. Now Suzan was a thrifty young lady and she decided to allow her parents to keep her money for her. In exchange they promised to **increase** her "account", on a weekly basis, by a fraction λ of the sum gathered on this account at the beginning of each week. Jack, on the other hand, decided to **spend** his money, but not all of it right away. He spent, also on a weekly basis, a fraction λ of the sum he still had at the beginning of each week. This description shows that Suzan's money M_S will increase with time, while Jack's money M_J will decrease in time. At the same time it is clear to see that both these processes **will not** be uniform. The weekly changes ΔM_S are not the same, since they depend on the changing value M_S itself; the same principle holds for M_J.

   Both these processes involve the fraction λ, which can be defined as:

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$$\lambda = \frac{\frac{\Delta M}{\Delta t}}{M},$$ (Δt denotes the passing of each week).

We can also consider this as a "formula for obtaining ΔM", analogically to equation (1):

$$\Delta M = \lambda \cdot M \cdot \Delta t \qquad (2)$$

Let's note that the fraction λ is not a number, but a quantity with units "one over week". In general, the unit of λ would be "one over the time unit chosen in the problem" – we have chosen Δt as one week. Let's also note that in Suzan's case the coefficient λ is positive, while in Jack's case it has a negative value. This corresponds to the difference between the growing value of M_S and the diminishing value of M_J. We usually want to emphasize this difference in formula (2). We therefore consider λ to always be positive, at the same time using the '+' or '–' signs in formula (2) as follows:

$$\Delta M_S = +\lambda \cdot M_S \cdot \Delta t \qquad (3a)$$

$$\Delta M_J = -\lambda \cdot M_J \cdot \Delta t \qquad (3b)$$

Formula 3a is used to calculate the weekly changes in the amount of Suzan's money and formula 3b applies to the weekly changes of Jack's money.

3. The simplest mathematical model of a process involves presenting the dependence of a quantity Q on a variable x. Such a dependence can be presented by means of a ready table or graph, a known function Q(x) or by an equation involving the (unknown) function Q(x), its initial value Q_0 and its change ΔQ associated to the change of its argument Δx and to parameters of the process. Such an equation can be used to fill in a table of values of the function Q on a step-by-step basis: starting with a given initial value of Q_0 for $x = x_0$ (usually the initial value of $x_0 = 0$), the value of ΔQ_0 can be calculated. We then set $Q(x_1) = Q_0 + \Delta Q_0$. The value of $x_1 = x_0 + \Delta x$, where Δx is the chosen increment of the argument x, usually the same for every step. Likewise, for any step (no. n+1) we assume that

$$Q(x_{n+1}) = Q(x_n) + \Delta Q_n \qquad (4)$$

Equations (1), (2), (3a) and (3b) are examples of calculating ΔQ_n.

In order to find how Susan's money increases in time, a 3-column spreadsheet can be used. The first column can contain time t (the consecutive weeks in each line

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being denoted as t_0, t_1, etc.), the second column can contain the value of the function M_S. The third column will then contain a formula (based on equation 3a) giving the weekly change ΔM_S. In the next line x should be increased by Δx and M_S should be increased by ΔM_S calculated in the previous line.

4. In mathematics we can define – using equation (2), (3a) or (3b) - the "exponential function". Let's consider the following property of a function f(x): equal changes Δx of its argument give changes Δf of its value proportional to the value f(x). Such a function is similar to an exponential function. The similarity gets better when the value of Δx becomes smaller. If this property holds for any (small) value of Δx:

$$\Delta f = \pm\lambda\cdot f(x)\cdot\Delta x \quad \text{or} \quad f(x + \Delta x) - f(x) = \pm\lambda\cdot f(x)\cdot\Delta x$$

then f(x) is an exponential function and is usually denoted by:

$$f(x) = F_0\cdot e^{\pm\lambda\cdot x} \qquad (4).$$

In this formula F_0 denotes the so-called initial value of the function. The irrational number e = 2,7182818459… is the „natural" base of the exponential function. The sign preceding the constant coefficient λ determines whether the function is an increasing one (the '+' sign) or a decreasing one (the '-' sign). In computer software and on calculators this function is usually denoted by EXP(x).

In our story about the twins, Suzan's money grows exponentially in time (approximately, since we have chosen one week for Δt, instead of a "moment" - the shortest possible time lapse). Such a growth is sometimes termed as "geometric progression". Jack's money declines exponentially in time (also approximately).

5. Get acquainted with the basic properties of an exponential function. Study graphs of functions of the following types:

- $f(x) = A\cdot e^{+\lambda\cdot x}$;
- $f(x) = A\cdot e^{-\lambda\cdot x}$;
- $f(x) = A\cdot (1 - e^{-\lambda\cdot x})$;
- $f(x) = A\cdot (e^{-\lambda\cdot x} - 1)$.

Consider using a spreadsheet in order to see the influence of each of the two (positive) parameters A and λ on the graph.

III. Problems in mathematics and banking

1. Consider the following statement: "We envisage three moments t_1, $t_2 > t_1$ and $t_3 > t_2$ fulfilling the requirement $t_2 - t_1 = t_3 - t_2$; if a function f is exponential, then $f(t_3)/f(t_2) = f(t_2)/f(t_1)$".

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Use a spreadsheet model prepared for Suzan's story to verify whether this statement holds for the function $M_S(t)$ for a chosen triplet (t_1; t_2; t_3). Use a similar spreadsheet and s new triplet, with another difference $t_2 - t_1$, to verify this statement, when applied to $M_J(t)$ in Jack's story. Write a short comment on the results you obtain.

2. Use both spreadsheet models to show that the consecutive values of M_S (respectively M_J) form a geometric sequence. Express the common ratio of such a sequence using λ.

3. Formula (3a) is commonly used to determine compound interest in bank accounts. In our story the *compounding frequency* is set to once per week, although it could be set to once per year or once per month. The constant parameter λ is closely related to the *interest rate*. Modify Suzan's spreadsheet to solve the following problem: "Three people open savings accounts with the same sum of money. Person A opens an account with a yearly compounding frequency at an interest rate of 1.56 % (yearly). Person B opens an account with a monthly compounding frequency at an interest rate of 0.13 % (monthly). Person C opens an account with a weekly compounding frequency.

   i) Which of the two persons, A or B, will have a higher amount of money after one year? Comment on your result after remarking that 1.56% = 12· 0.13%.

   ii) What interest rate should be set to person C's account to ensure the same amount of money after one year as in the case of person A. Comment on your result after remarking that 1.56% = 52· 0.03%.".

4. Jack was given 1000 gcu for his birthday (gcu is a **g**eneral **c**urrency **u**nit). He decided to spend 5% of his belongings each week. One fine day Jack noticed that his belongings had shrunken to 100 gcu. How long was that after his birthday? Comment on the precision of your result.

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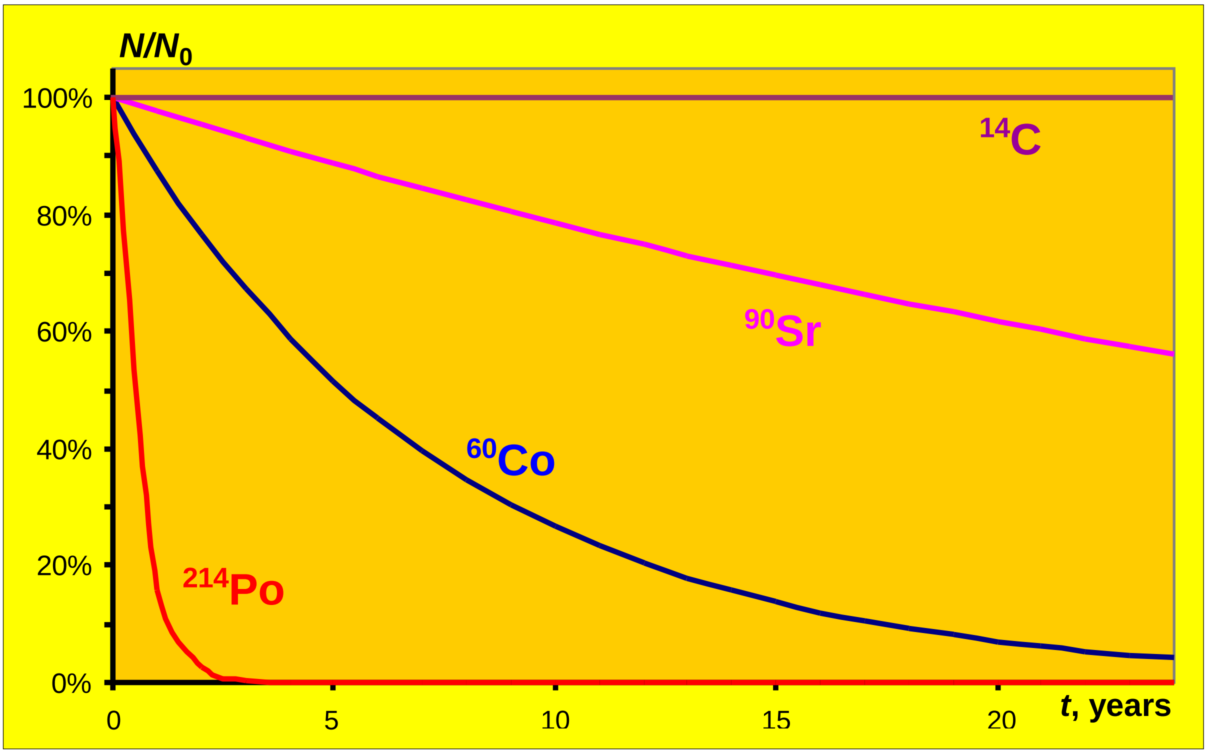
IV. Problems in physics

Get acquainted with the phenomenon of radioactive decay in nuclear physics.

1. The graph below shows the decay process of four different isotopes, all of them radioactive. Explain the different shapes of the four lines on the graph. Each of the isotopes is characterized by its decay constant λ, and the quantity N of non-decayed nuclei is given by the exponential function:

$$N(t) = N_0 \cdot e^{-\lambda \cdot t}$$

with N_0 standing for the initial amount of radioactive nuclei.



2. Every exponential decrease is characterized by its half-life, denoted as $t_{1/2}$. Use a table of values or a graph of an exponential function with a given value of λ to find $t_{1/2}$. Change λ and verify the hypothesis that the product $\lambda \cdot t_{1/2}$ is constant. Find its value and its units.

3. Show that the time interval $t_{1/2} = t_2 - t_1$ is the same between two random moments, in which $N(t_2) = ½ \cdot N(t_1)$.

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4. Instead of the half-life, any “q-life” (denoted by t_q) could be used to characterize a radioactive decay, with q being a fraction between zero and one: $0 < q < 1$. Choose any fraction q greater than ½ and use the table or graph to show that $N(t + t_q)/N(t) = q$ for two different values of t.

Get acquainted with the process of charging a capacitor through a resistor by a DC power source.

1. Assume that the dependence on time of charge Q accumulated in a capacitor of constant capacitance C, with initial charge equal to zero, being charged through a resistor of constant resistance R by a DC power source of constant voltage U is given by the exponential function:

$$Q(t) = Q_m \cdot (1 - e^{-\lambda \cdot t})$$

with Q_m standing for the maximum charge attained after “infinite” charging. State the dependence of λ and Q_m on C, R and U.

2. Define the charging half-time $t_{1/2}$, a quantity analogical to the half-life of an isotope in nuclear physics. State the dependence of $t_{1/2}$ on C, R and U.
3. Find the dependence between $Q(t_1)$ and $Q(t_2)$ when t_1 and t_2 satisfy $t_2 - t_1 = t_{1/2}$.
4. Use a spreadsheet program to verify the following statement: the energy accumulated in a fully-charged capacitor is always equal to one-half of the work done by the power source in the charging process, independently of the value of R.

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V. Problems in mathematical modeling

1. Evolution of a strain of bacteria.

It is commonly assumed that in ideal conditions a strain of bacteria will evolve exponentially in time, since an individual bacterium divides into two after a certain time t_d, constant for the strain. This means that the number N of bacteria is given by an exponential function:

$$N(t) = N_0 \cdot e^{\lambda \cdot t}$$

with N_0 standing for the initial amount of bacteria.

a) State the relation between time t_d and the constant λ in the above equation.

In real life conditions are seldom ideal.

b) Use a spreadsheet to elaborate a mathematical model showing the evolution of a strain of bacteria in a finite biotope. Comment on any resemblance (or its lack) of the obtained result to an exponential function.

c) Use a spreadsheet to elaborate a mathematical model showing the evolution of a strain of bacteria in a biotope into which a bacteriophage is introduced at a certain stage of the evolution. Comment on any resemblance (or its lack) of the obtained result to an exponential function.

2. Evolution of a set of two reactants in a chemical reaction.

It can be assumed that two chemical reactants react according to the following law: in a given time interval the decrease in chemical concentration (denoted by C_1 and C_2) of each reactant is proportional to the product of their momentary concentrations and the time interval; furthermore the changes in C_1 and C_2 are at all times equal.

a) Use a spreadsheet to elaborate a mathematical model showing the time evolution of the concentrations in such a case. Verify whether any one of C_1 or C_2 the concentrations diminishes following an exponential function.

b) Try changing the difference D between the initial values of C_1 and C_2. Does this make any one of C_1 or C_2 approach an exponential function?

c) Modify your model to account for a change in reaction speed due to conditions for the reaction becoming more favorable as a result of the reaction itself (e.g. a rise in temperature in an exothermic reaction).

3. Evolution of the number of soldiers in two armies fighting a battle.

Envisage two armies, initially consisting of N_{01} and N_{02} soldiers respectively, engaged in a battle. Assume that in a given time interval the number of soldiers put *hors de*

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combat in one army is proportional to the momentary number of soldiers in the other army and to the time interval itself. The two proportionality coefficients describe many factors, some of them imponderable, like the soldiers' morale, and are very unlikely to be equal.

a) Use a spreadsheet to elaborate a mathematical model showing the time evolution of the number of fighting soldiers in both armies. Show that in a general case neither N_1 nor N_2 diminishes exponentially.

b) The functions $N_1(t)$ and $N_2(t)$ may be exponential under certain conditions. Find a combination of values of the two proportionality coefficients which gives such a result. Show that such a combination depends on N_{01} and N_{02}.

c) Let's imagine that army 1, initially counting N_{01} soldiers, can make a "draw" with army 2, initially counting N_{02} soldiers. A draw means that the number of fighting soldiers in both armies approaches zero at the same time. Now the commander of army 1 could decide to split his force into two detachments and fight two battles: the first one ends when all the soldiers of the first detachment are put *hors de combat*; the second battle engages the second detachments with the remnants of army 2. Show that such a decision is unfavorable. It is even said that if two detachments are to be worth one army, the sum of the squares of the initial numbers of soldiers in the detachments should be equal to the square of the initial number of soldiers in the whole army. Verify whether such a conviction is true.

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VI. Problems in experimental data analysis

1. Time dependence of the temperature of a corps cooling in constant room temperature.

A small piece of molten metal solidifies at 600 ^{0}C, and then cools down at constant room temperature T_r = 20 ^{0}C. The metal's temperature T was measured in 15-second intervals, starting at 500 ^{0}C. The results are given in the table below. All uncertainties were small enough as not to affect the last digit shown in the table.

t (s)	T (^{0}C)	t (s)	T (^{0}C)	t (s)	T (^{0}C)	t (s)	T (^{0}C)
0	500,0	120	74,6	225	33,0	330	23,2
15	323,1	135	64,3	240	30,6	345	22,6
30	242,0	150	55,9	255	28,7	360	22,1
45	189,8	165	49,3	270	27,1	375	21,8
60	152,8	180	43,8	285	25,8	390	21,4
75	125,2	195	39,5	300	24,8	405	21,2
90	104,1	210	35,9	315	23,9	420	21,0
105	87,6						

a) Verify whether or not the data fit an exponential function.

b) Comment on the precision of your result.

c) Give any possible reason(s), related to the process of cooling, why your result is appropriate.

2. Time dependence of voltage on an electrolytic capacitor being discharged through a resistor.

An electrolytic capacitor (nominally C = 5 mF, V_{max} = 15 V) is being discharged through a resistor (nominally R = 1 kΩ). The momentary voltage V on the capacitor is measured at 0,5-second intervals, starting at 12 V. The results are given in the table below. All uncertainties were small enough as not to affect the last digit shown in the table.

t (s)	U (V)	t (s)	U (V)	t (s)	U (V)	t (s)	U (V)

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0,0	12,00	4,5	4,72	9,0	1,78	13,5	0,58
0,5	10,83	5,0	4,24	9,5	1,59	14,0	0,50
1,0	9,77	5,5	3,82	10,0	1,41	14,5	0,42
1,5	8,81	6,0	3,43	10,5	1,26	15,0	0,36
2,0	7,95	6,5	3,08	11,0	1,11	15,5	0,29
2,5	7,16	7,0	2,77	11,5	0,99	16,0	0,24
3,0	6,46	7,5	2,48	12,0	0,87	16,5	0,18
3,5	5,82	8,0	2,22	12,5	0,76	17,0	0,13
4,0	5,24	8,5	1,99	13,0	0,67		

a) Verify whether or not the data fit an exponential function.

b) Comment on the precision of your result.

c) Give any possible reason(s) , related to the process of discharging a capacitor, why your result is appropriate.

3. Dependence of the tension force in one of two identical cords upholding a weight on the angle between the cord and the vertical direction.

A weight is suspended by means of two identical cords. The free end of each cord can be attached at any point of a ring situated in a vertical plane, in order to position the weight at the center of the ring. The angle α was increased in steps of 5^0 and the corresponding tension force F in each of the cords was measured. The results are given in the table. All uncertainties were small enough as not to affect the last digit shown in the table.

α (deg)	F (N)
5	36,1
10	36,6
15	37,3

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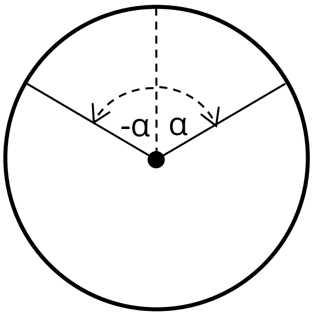

20	38,3
25	39,7
30	41,6
35	43,9
40	47,0
45	50,9
50	56,0
55	62,8
60	72,0
65	85,2
70	105,3
75	139,1
80	207,3

a) Verify whether or not the data fit an exponential function.

b) Comment on the precision of your result.

c) Give any possible reason(s), related to the process of tension distribution in statics, why your result is appropriate.

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